Generalization of Banach Contraction Mapping Principle and Fixed Point Theorem by Altering Distances between the Points

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ABSTRACT *The concept of fixed point opens a door of vast applications of Mathematics in many fields like Economics, Game Theory etc. Existence of fixed point becomes a centre of attraction for many researchers in applied Mathematics. Stefan Banach proved a first fixed point theorem in the settings of Metric Spaces. But the result requires the continuity of the self map. In this research paper we generalize the Banach Contraction Mapping Principle. This theorem uses a function from the set of all positive real numbers to itself to define a contractive condition. We have also presented an example to prove that the theorem indeed exists. Further we have established a fixed point theorem in complete metric space by using altering distances between the points. This result generalizes the fixed point theorems of M. S. Khan et al., Reich and Rakotch.*

Key words: Complete Metric Space, Fixed Point, Altering Distance Function.

1. Introduction

Stefan Banach, a celebrated Mathematician from Poland proved the first fixed point theorem in 1922, known as the "Banach Contraction Mapping Principle" [Banach, 1922]. Kannan invented new type of contractions called Kannan Mappings. Kannan proved that his contractions are independent of Banach contractions and also proved that every Kannan mapping on a complete metric space has a unique fixed point [Kannan, 1969]. There are numerous extensions of Banach Contraction Mapping Principle in the literature. See [Ciric, 2003, Jachymski, 1997, Meer and Keeler, 1969]. Some of the important are Boyd and Wong [Boyd and Wong, 1969] and Matkowski [Matkowski, 1975]. We have proved a fixed point theorem on a complete metric space that shows a close resemblance to the theorems of Boyd, Wong and Matkowski. The next generalization of Banach Contraction Mapping Principle is in terms of altering distances function. Delbosco [Delbosco, 1976-77] and Skof [Skof, 1977] established fixed point theorem for self maps of complete metric spaces by altering the distances between the points. Later on F. Skof, M. S. Khan, M. Swaleh and S. Sessa [F. Skof, M. S. Khan, M. Swaleh and S. Sessa, 1984] studied the fixed point theorems regarding the altering the distances between the points. Sastry K. P. R., Babu G. V. R. also proved the fixed point theorems about the altering distances between the points [Babu, G. V. R., 2001, Sastry, K. P. R., 1999]. Jha K. and Pant R. P. worked further in this direction [Jha K. and Pant R. P., 2002-2007]. The purpose of this paper is to study a fixed point theorem by altering distances between the points.

2. Preliminaries and Definitions

Definition 2.1 (Metric Space) [Kreyszig, 1989]**:** A "Metric Space" is a pair (X,d), where X is a set and d is a metric on X (or distance function on X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(M1) d is real-valued, finite and non-negative, (M2) $d(x,y)=0$ if and only if $x = y$,

(M3) $d(x,y)=d(y,x)$ (Symmetry)

(M4) $d(x,y)\le d(x,z)+d(z,y)$(Triangle Inequality)

Example 2.1 [Kreyszig, 1989]: The set of all real numbers, taken with the usual metric defined by $d(x,y)=|x-y|$ is a metric space.

Note 2.1 [Goldberg, 1970**]:** It is important to note that if (X,d) is a metric space and $A\subseteq X$, then (A,d) is also a metric space.

Definition 2.2: A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X,d) is said to converge or to be convergent if there is an $x\in X$ such that $\lim_{n\to\infty} d(x_n,x)=0$, x is called the limit of $\{x_n\}_{n=1}^{\infty}$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n\to\infty$.

Definition 2.3: A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X,d) is said to be a "Cauchy Sequence" if for every $\varepsilon>0$ there is an $N=N(\varepsilon)$ such that $d(x_m,x_n)<\varepsilon$ for every $m,n\ge N$

Theorem 2.1: Every convergent sequence in a metric space is a Cauchy sequence.

Note 2.2: The converse of the above theorem is not true in general. That is a Cauchy sequence in a metric space X may or may not converge in X .

Definition 2.4: A metric space (X,d) is said to be a "Complete Metric Space" if every Cauchy Sequence in X converges in X.

Definition 2.5: A "Fixed Point" of a mapping $T:X\to X$ is an $x\in X$ which is mapped onto itself, that is $Tx=x$.

Definition 2.6: Let (X,d) be a metric space and let T be a mapping on X. Then T is called a "Contraction" if there exists $r\in[0,1)$ such that $d(Tx,Ty)\le r d(x,y)$ for all $x,y\in X$.

Theorem 2.2 [Banach 1922]: Let (X,d) be a complete metric space and let T be a contraction on X . Then T has a unique fixed point.

Theorem 2.3 [Boyd–Wong Theorem, 1969]: Let (X,d) be a complete metric space, and suppose that $T:X\to X$ satisfies $d(Tx,Ty)\le\psi(d(x,y))$, for all $x,y\in X$, where $\psi:\mathbb{R}\to[0,\infty)$ is upper semi-continuous from the right (that is, for any sequence $t_n\downarrow t\ge 0 \Rightarrow \limsup_{n\to\infty}\psi(t_n)\le\psi(t)$) and satisfies $0<\psi(t)\le t$ for $t>0$. Then, T has a unique fixed point.

Theorem 2.4 [Matkowski. 1975]: Let (X,d) be a complete metric space and suppose that $T:X\to X$ satisfies $d(Tx,Ty)\le\psi(d(x,y))$ for all $x,y\in X$, where, $\psi:(0,\infty)\to(0,\infty)$ is monotone non-decreasing and satisfies $\lim_{n\to\infty}\psi^n(t)=0$ for all $t>0$. Then T has a unique fixed point in X .

Definition 2.8 [Gupta, 2016]: Let Ψ be the set of all functions $\phi:[0,\infty)\to[0,\infty)$ satisfying the following conditions:

(1) $\phi(t)$ is strictly increasing and continuous.

(2) $\phi(t)=0$ if and only if $t=0$.

Such functions are called "Altering Distance Functions"

Definition 2.9 [Khan, Swaleh, Sessa, 1984]: Let Φ be the set of all functions $\phi:[0,\infty)\to[0,\infty)$ satisfying the conditions (1) and (2) and the following condition: (3) $\phi(t)\ge M.t^{\mu}$ for every $t>0$, where $M>0,\mu>0$ are constants.

Definition 2.10: Let Ω be the se of all functions $\phi:[0,\infty)\to[0,\infty)$ satisfying the conditions (1) and (2) and the following condition: (4) $\phi(a+b)\le\phi(a)+\phi(b)$.

Example 2.2: The following are examples of the functions in the set Ω defined in the definition 2.10.

1) $\phi(t)=t,\qquad \forall t\in[0,\infty)$

2) $\phi(t)=\tanh t, \quad \forall t\in[0,\infty)$

3) $\phi(t)=\tan^{-1} t, \quad \forall t\in[0,\infty)$

4) $\phi(t)=\sec^{-1} t, \quad \forall t\in[0,\infty)$

Theorem 2.5 [Khan, Swaleh, Sessa, 1984]: Let (X,d) be a complete metric space, let $\phi\in\Psi$ and let $T:X\to X$ be a self mapping which satisfies the inequality $\phi\left(d(Tx,Ty)\right)\le c\phi\left(d(x,y)\right)$ for all $x,y\in X$ and for some $0<c<1$. Then T has a unique fixed point.

Theorem 2.6 [Skof, 1977]: Let T be a self map of a complete metric space (X,d) and let $\phi\in\Psi$ satisfying for every $x,y\in X$,

$$\phi\left(d(Tx,Ty)\right)\le a\phi(d(x,y))+b\phi\left(d(x,Tx)\right)+c\phi d\left((y,Ty)\right)$$

for all $x,y\in X$, where $0<a+b+c<1$ Then T has a unique fixed point.

Theorem 2.7 [Khan, Swaleh, Sessa, 1984]: Let (X,d) be a complete metric space, T a self map of X and $\phi\in\Psi$. Furthermore, let a,b,c be three decreasing functions from $\mathbb{R}^+\setminus\{0\}$ into $[0,1)$ such that $a+2b+c<1$. Suppose that T satisfies the following condition:

$$\phi\left(d(Tx,Ty)\right)\le a\phi\left(d(x,y)\right)+b\begin{pmatrix}\phi(d(x,Tx))+\\ \phi(d(y,Ty))\end{pmatrix}+c\min\{\phi(d(x,Ty)),\phi(d(y,Tx))\}$$

where, $x,y\in X$ and $x\neq y$. Then T has a unique fixed point.

3. Main Results

Theorem 3.1: Let (X,d) be a complete metric space and suppose that $T:X\to X$ satisfies

$$\text{(A)}\quad \begin{aligned} d(Tx,Ty)\le{}& \alpha\psi\left(d(x,Tx)\right)+\beta\psi\left(d(y,Ty)\right)+\\ &\gamma\psi\left(d(x,y)\right)+\delta\psi\left(d(x,Ty)\right)+\\ &\theta\psi\left(d(y,Tx)\right)\end{aligned}$$

for all $x,y\in X$, where $\psi:\mathbb{R}\to[0,\infty)$ satisfies $0\le\psi(t)<t$ for all $t>0$, $\psi(0)=0$

Also

$$0<\alpha+\beta+\gamma+2\delta+\theta<1,$$
$$\alpha>0,\beta>0,\gamma>0,\delta>0,\theta>0.$$

Then T has a unique fixed point in X.

Proof: Let $x_0\in X$ be an arbitrary but a fixed element in X. Define a sequence of iterates $\{x_n\}_{n=1}^{\infty}$ in X by

$$x_1=Tx_0, x_2=Tx_1=T^2x_0, x_3=Tx_2=T^3x_0,\ldots\ldots, x_n=Tx_{n-1}=T^nx_0,\ldots\ldots$$

By the condition (A) on T we get,

$$\begin{aligned} d(x_n,x_{n+1})&=d(Tx_{n-1},Tx_n)\\ &\le\alpha\psi\left(d(x_{n-1},Tx_{n-1})\right)+\beta\psi\left(d(x_n,Tx_n)\right)+\\ &\quad\gamma\psi\left(d(x_{n-1},x_n)\right)+\delta\psi\left(d(x_{n-1},Tx_n)\right)+\theta\psi\left(d(x_n,Tx_{n-1})\right)\\ &=\alpha\psi\left(d(x_{n-1},x_n)\right)+\beta\psi\left(d(x_n,x_{n+1})\right)+\\ &\quad\gamma\psi\left(d(x_{n-1},x_n)\right)+\delta\psi\left(d(x_{n-1},x_{n+1})\right)+\theta\psi\left(d(x_n,x_n)\right)\\ &<\alpha d(x_{n-1},x_n)+\beta d(x_n,x_{n+1})+\\ &\quad\gamma d(x_{n-1},x_n)+\delta\left(d(x_{n-1},x_{n+1})\right)+0 \qquad (\because\psi(t)<t)\end{aligned}$$

Thus,

$$\begin{aligned} d(x_n,x_{n+1})<{}&\alpha d(x_{n-1},x_n)+\beta d(x_n,x_{n+1})+\\ &\gamma d(x_{n-1},x_n)+\delta\left(d(x_{n-1},x_n)\right)+\\ &\delta\left(d(x_n,x_{n+1})\right)\end{aligned}$$

where, $h=\dfrac{\alpha+\gamma+\delta}{1-\beta-\delta}$. Here $0<h<1$.

$$\therefore(1-\beta-\delta)d(x_{n+1},x_n)<(\alpha+\gamma+\delta)d(x_{n-1},x_n)$$

$$\therefore d(x_{n+1},x_n)<\frac{\alpha+\gamma+\delta}{1-\beta-\delta}d(x_{n-1},x_n)$$

$$\therefore d(x_{n+1},x_n)<hd(x_{n-1},x_n)$$

Continuing in this way, we get $d(x_{n+1},x_n)<h^nd(x_0,x_1)$. Taking limit as $n\to\infty$ we get,

$d(x_{n+1}, x_n) \to 0 \qquad (\because 0 < h < 1)$

Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. As X is a complete metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. We shall show that x is a fixed point of T.

As T a is continuous function we have,

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T\left(\lim_{n\to\infty} x_{n-1}\right) = Tx .$$

Therefore $Tx = x$ and x is a fixed point of T. Next we shall show that x is unique fixed point of T.

Let $y \in X$ be another fixed point of T.

Again by the condition (A) we get,

$$\begin{aligned} d(x,y) = d(Tx,Ty) &\le \alpha\psi\left(d(x,Tx)\right) + \\ &\beta\psi\left(d(y,Ty)\right) + \gamma\psi\left(d(x,y)\right) + \\ &\delta\psi\left(d(x,Ty)\right) + \theta\psi\left(d(y,Tx)\right) \end{aligned}$$

$$\begin{aligned} &= \alpha\psi\left(d(x,x)\right) + \beta\psi\left(d(y,y)\right) + \gamma\psi\left(d(x,y)\right) + \\ &\quad \delta\psi\left(d(x,y)\right) + \theta\psi\left(d(y,x)\right) \\ &= \gamma\psi\left(d(x,y)\right) + \delta\psi\left(d(x,y)\right) + \\ &\quad \theta\psi\left(d(x,y)\right) \\ &= (\gamma + \delta + \theta) d(x,y) \\ &< d(x,y) \qquad (\because \gamma + \delta + \theta < 1.) \end{aligned}$$

Thus, $d(x,y) < d(x,y)$

$$\begin{aligned} &= \alpha\psi\left(0\right) + \beta\psi(0) + \gamma\psi\left(d(x,y)\right) + \\ &\quad \delta\psi\left(d(x,y)\right) + \theta\psi\left(d(y,x)\right) \end{aligned}$$

This is possible if and only if $d(x,y) = 0$. Thus $x = y$.

Example 3.1: Consider the complete metric space of all non-negative real numbers with absolute value metric. Suppose that $T : X \to X$ defined by $Tx = \frac{x}{16}$. Let $\psi : \mathbb{R} \to [0,\infty)$ be defined by $\psi(t) = \frac{t}{2}$. The function $\psi(t)$ is continuous, also $0 < \psi(t) < t$ for all $t > 0$, $\psi(0) = 0$. Let

$$\alpha = \beta = \gamma = \delta = \theta = \frac{1}{8}.$$

Then clearly

$$0 < \alpha + \beta + \gamma + 2\delta + \theta = \frac{3}{4} < 1,$$

$$\alpha > 0, \beta > 0, \gamma > 0, \delta > 0, \theta > 0.$$

We verify that the condition (A) of the theorem 3.1 is satisfied.

We observe that

$$d(Tx,Ty) = d\left(\frac{x}{16}, \frac{y}{16}\right) = \frac{|x-y|}{16}.$$

Also

$$\begin{aligned} &\alpha\psi\left(d\left(x,Tx\right)\right) + \beta\psi\left(d(y,Ty)\right) + \\ &\gamma\psi\left(d(x,y)\right) + \delta\psi\left(d(x,Ty)\right) + \\ &\theta\psi\left(d(y,Tx)\right) \\ &= \frac{1}{8}\psi\left(d\left(x, \frac{x}{16}\right)\right) + \frac{1}{8}\psi\left(d\left(y, \frac{y}{16}\right)\right) + \\ &\frac{1}{8}\psi\left(d(x,y)\right) + \frac{1}{8}\psi\left(d\left(x, \frac{y}{16}\right)\right) + \frac{1}{8}\psi\left(d\left(y, \frac{x}{16}\right)\right) \\ &= \frac{1}{8}\psi\left(\frac{15x}{16}\right) + \frac{1}{8}\psi\left(\frac{15y}{16}\right) + \frac{1}{8}\psi\left(|x-y|\right) + \\ &\frac{1}{8}\psi\left(\frac{|16x-y|}{16}\right) + \frac{1}{8}\psi\left(\frac{|16y-x|}{16}\right) \\ &= \frac{1}{8}\frac{\left(\frac{15x}{16}\right)}{2} + \frac{1}{8}\frac{\left(\frac{15y}{16}\right)}{2} + \frac{1}{8}\frac{|x-y|}{2} + \frac{1}{8}\frac{\left(\frac{|16x-y|}{16}\right)}{2} + \\ &\frac{1}{8}\frac{\left(\frac{|16y-x|}{16}\right)}{2} \\ &= \frac{15x}{256} + \frac{15y}{256} + \frac{|x-y|}{16} + \frac{|16x-y|}{256} + \frac{|16y-x|}{256} \\ &= \frac{15(x+y)}{256} + \frac{|x-y|}{16} + \frac{|16x-y|}{256} + \frac{|16y-x|}{256}. \end{aligned}$$

Thus clearly

$$d(Tx,Ty) = d\left(\frac{x}{16},\frac{y}{16}\right) = \frac{|x-y|}{16} < \frac{15(x+y)}{256} + \frac{|x-y|}{16} + \frac{|16x-y|}{256} + \frac{|16y-x|}{256}$$
$$= \alpha\psi\left(d\left(x,Tx\right)\right) + \beta\psi\left(d(y,Ty)\right) + \gamma\psi\left(d(x,y)\right) + \delta\psi\left(d(x,Ty)\right) + \theta\psi\left(d(y,Tx)\right)$$

for all $x \in \mathbb{R}^+$. The condition (A) of the theorem 3.1 is satisfied. We see that $x = 0$ is the unique fixed point of the function T.

Theorem 3.2: Let (X,d) be a complete metric space and T be a self map of X. Let $\phi \in \Omega$. Furthermore let T satisfy the condition:

$$\text{(B)}\quad \phi\left(d(Tx,Ty)\right) \le \alpha\left(d(x,y)\right)\phi\left(d(x,Tx)\right) + \beta\left(d(x,y)\right)\phi\left(d(y,Ty)\right) + \gamma\left(d(x,y)\right)\phi\left(d(x,y)\right) + \delta\left(d(x,y)\right)\phi\left(d(x,Ty)\right) + \theta\left(d(x,y)\right)\phi\left(d(y,Tx)\right)$$

for all $x, y \in X, x \neq y$, where $\alpha,\beta,\gamma,\delta,\theta > 0$ are decreasing functions from $\mathbb{R}^+ \setminus \{0\}$ into $[0,1)$, such that $\alpha+\beta+\gamma+2\delta+\theta<1$. Then T has a unique fixed point in X.

Proof: For the sake of simplicity we denote $\alpha\left(d(x,y)\right)$ by α, $\beta\left(d(x,y)\right)$ by β, $\gamma\left(d(x,y)\right)$ by γ, $\delta\left(d(x,y)\right)$ by δ and $\theta\left(d(x,y)\right)$ by θ.

Let x_0 be a point in X. We define $x_{n+1} = Tx_n$ for $n = 0,1,2,3.....$.

We shall show that T has fixed point.

From the condition (B) on T we have,

$$\phi\left(d(x_n,x_{n+1})\right) = \phi\left(d(Tx_{n-1},Tx_n)\right)$$
$$\le \alpha\phi\left(d(x_{n-1},Tx_{n-1})\right) + \beta\phi\left(d(x_n,Tx_n)\right) + \gamma\phi\left(d(x_{n-1},x_n)\right) + \delta\phi\left(d(x_{n-1},Tx_n)\right) + \theta\phi\left(d(x_n,Tx_{n-1})\right)$$
$$= \alpha\phi\left(d(x_{n-1},x_n)\right) + \beta\phi\left(d(x_n,x_{n+1})\right) + \gamma\phi\left(d(x_{n-1},x_n)\right) + \delta\phi\left(d(x_{n-1},x_{n+1})\right) + \theta\phi\left(d(x_n,x_n)\right)$$
$$\le \alpha\phi\left(d(x_{n-1},x_n)\right) + \beta\phi\left(d(x_n,x_{n+1})\right) + \gamma\phi\left(d(x_{n-1},x_n)\right) + \delta\phi\left(d(x_{n-1},x_n) + d(x_n,x_{n+1})\right)$$
$$\le \alpha\phi\left(d(x_{n-1},x_n)\right) + \beta\phi\left(d(x_n,x_{n+1})\right) + \gamma\phi\left(d(x_{n-1},x_n)\right) + \delta\phi\left(d(x_{n-1},x_n)\right) + \delta\phi\left(d(x_n,x_{n+1})\right)$$

Hence we obtain

$$\phi\left(d(x_n,x_{n+1})\right) \le \frac{\alpha+\gamma+\delta}{1-\beta-\delta}\phi\left(d(x_{n-1},x_n)\right) < \phi\left(d(x_{n-1},x_n)\right) \qquad (1)$$

Thus $\phi\left(d(x_n,x_{n+1})\right) < \phi\left(d(x_{n-1},x_n)\right)$. Now ϕ is an increasing function. Therefore we have $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$ for all $n = 1,2,3,.....$ .Therefore $\{d(x_n,x_{n+1})\}_{n=0}^{\infty}$ is a decreasing sequence which is also bounded below (by 0). Let $\lim_{n\to\infty} d(x_n,x_{n+1}) = L$. In equation (1) taking limit as $n \to \infty$ we get

$$\lim_{n\to\infty}\phi\left(d(x_n,x_{n+1})\right) < \lim_{n\to\infty}\phi\left(d(x_{n-1},x_n)\right)$$
$$\therefore \phi\left(\lim_{n\to\infty} d(x_n,x_{n+1})\right) < \phi\left(\lim_{n\to\infty} d(x_{n-1},x_n)\right)$$
$$\therefore \phi(\mathrm{L}) < \phi(\mathrm{L})$$
$$\therefore \mathrm{L} < L$$
$$\therefore L = 0.$$

Now we shall prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Suppose $\{x_n\}_{n=1}^{\infty}$ is not a Cauchy sequence. Thus there exists $\varepsilon > 0$ and two sequences $\{r_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ such that for every $n \in \mathbb{N} \cup \{0\}$, we find that

$r_n > t_n \geq 0, d(x_{r_n}, x_{t_n}) \geq \varepsilon$ and $d(x_{r_n-1}, x_{t_n}) < \varepsilon.$

For each $n \geq 0$ we put $S_n = d(x_{r_n}, x_{t_n})$. Then we have

$$\varepsilon \leq S_n \leq d(x_{r_n-1}, x_{r_n}) + d(x_{r_n-1}, x_{t_n}) < d(x_{r_n-1}, x_{r_n}) + \varepsilon$$

Since $\{d(x_n, x_{n+1})\}_{n=1}^{\infty}$ converges to 0, $\{S_n\}_{n=1}^{\infty}$ converges to ε.

Furthermore triangle inequality implies for each $n \geq 0$,

$$-d(x_{r_n}, x_{r_n+1}) - d(x_{t_n}, x_{t_n+1}) + d(x_{r_n}, x_{t_n})$$
$$\leq d(x_{r_n+1}, x_{t_n+1}) \leq d(x_{r_n}, x_{r_n+1}) + d(x_{t_n}, x_{t_n+1}) + S_n$$

This implies $\left\{d(x_{r_n+1}, x_{t_n+1})\right\}_{n=1}^{\infty}$ converges to ε.

From the condition (B) we have,

$$\phi\left(d(x_{r_n+1}, x_{t_n+1})\right) = \phi\left(d(Tx_{r_n}, Tx_{t_n})\right)$$
$$\leq \alpha\phi\left(d(x_{r_n}, Tx_{r_n})\right) + \beta\phi\left(d(x_{t_n}, Tx_{t_n})\right) +$$
$$\gamma\phi\left(d(x_{r_n}, x_{t_n})\right) + \delta\phi\left(d(x_{r_n}, Tx_{t_n})\right) +$$
$$\theta\phi\left(d(x_{t_n}, Tx_{r_n})\right)$$
$$= \alpha\phi\left(d(x_{r_n}, x_{r_n+1})\right) + \beta\phi\left(d(x_{t_n}, x_{t_n+1})\right) +$$
$$\gamma\phi\left(d(x_{r_n}, x_{t_n})\right) + \delta\phi\left(d(x_{r_n}, x_{t_n+1})\right) +$$
$$\theta\phi\left(d(x_{t_n}, x_{r_n+1})\right)$$
$$\leq \alpha\phi\left(d(x_{r_n}, x_{r_n+1})\right) + \beta\phi\left(d(x_{t_n}, x_{t_n+1})\right) +$$
$$\gamma\phi\left(d(x_{r_n}, x_{t_n})\right) + \delta\phi\left(d(x_{r_n}, x_{t_n}) + d(x_{t_n}, x_{t_n+1})\right) +$$
$$\theta\phi\left(d(x_{t_n}, x_{t_n+1}) + d(x_{t_n+1}, x_{r_n+1})\right)$$

As $n \to \infty$ we get $\phi(\varepsilon) \leq (\gamma + \delta + \theta)\phi(\varepsilon) < \phi(\varepsilon)$ which is a contradiction. Therefore $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence. By the completeness of the metric space X, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to some point $z \in X$. Now we show that z is a fixed point of T. We put $\rho_n = d(z, x_n)$.

We have

$$\phi\left(d(x_{n+1}, Tz)\right) = \phi\left(d(Tx_n, Tz)\right)$$
$$\leq \alpha\phi\left(d(x_n, Tx_n)\right) + \beta\phi\left(d(z, Tz)\right) +$$
$$\gamma\phi\left(d(x_n, z)\right) + \delta\phi\left(d(x_n, Tz)\right) +$$
$$\theta\phi\left(d(z, Tx_n)\right)$$
$$= \alpha\phi\left(d(x_n, x_{n+1})\right) + \beta\phi\left(d(z, Tz)\right) +$$
$$\gamma\phi\left(d(x_n, z)\right) + \delta\phi\left(d(x_n, Tz)\right) +$$
$$\theta\phi\left(d(z, x_{n+1})\right)$$
$$< \phi\left(d(x_n, x_{n+1})\right) + \beta\phi\left(d(z, Tz)\right) +$$
$$\phi\left(d(x_n, z)\right) + \frac{1}{2}\phi\left(d(x_n, Tz)\right) +$$
$$\phi\left(d(z, x_{n+1})\right)$$

Taking limit as $n \to \infty$ we get,

$$\limsup_{n\to\infty} \phi\left(d(x_{n+1}, Tz)\right) < \left(\beta + \frac{1}{2}\right)\phi\left(d(\mathrm{z}, \mathrm{Tz})\right) \quad (2)$$

On the other hand we also have by triangle inequality,

$$d(z, Tz) \leq d(z, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tz)$$
$$\therefore \phi\left(d(z, Tz)\right) \leq \phi\begin{pmatrix} d(z, x_n) + d(x_n, x_{n+1}) + \\ d(x_{n+1}, Tz) \end{pmatrix}$$

This implies that

$$\phi\left(d(z, Tz)\right) \leq \limsup_{n\to\infty} \phi\left(d(x_{n+1}, Tz)\right)$$

Thus (2) and (3) imply

$$\limsup_{n\to\infty} \phi\left(d(x_{n+1}, Tz)\right) < \left(\beta + \frac{1}{2}\right)\phi\left(d(z, Tz)\right) < \phi\left(d(z, Tz)\right) \leq \limsup_{n\to\infty} \phi\left(d(x_{n+1}, Tz)\right)$$

Thus we must have $\phi\left(d(z, Tz)\right) = 0$. So $Tz = z$. That is, z is a fixed point of T.

Suppose T has another fixed point as $y \in X$. Then

$$\phi\left(d(x,y)\right)=\phi\left(d(Tx,Ty)\right)\le\alpha\phi\left(d(x,Tx)\right)+\beta\phi\left(d(y,Ty)\right)+\gamma\phi\left(d(x,y)\right)+\delta\phi\left(d(x,Ty)\right)+\theta\phi\left(d(y,Tx)\right)$$
$$=\alpha\phi\left(d(x,x)\right)+\beta\phi\left(d(y,y)\right)+\gamma\phi\left(d(x,y)\right)+\delta\phi\left(d(x,y)\right)+\theta\phi\left(d(y,x)\right)$$
$$=\gamma\phi\left(d(x,y)\right)+\delta\phi\left(d(x,y)\right)+\theta\phi\left(d(x,y)\right)=(\gamma+\delta+\theta)\phi\left(d(x,y)\right)<\phi\left(d(x,y)\right)$$
$$(\because \gamma+\delta+\theta<1)$$

Thus $\phi\left(d(x,y)\right)<\phi\left(d(x,y)\right)$. This is possible if $\phi\left(d(y,x)\right)=0$. Thus $x=y$. Thus the fixed point of T is unique. This completes the proof.

Note 3.1: We make a note of it that the continuity of T is not needed in the theorem 3.2.

If we take $\alpha=\beta=\gamma=$ constant and $\delta=\theta=0$ in the theorem 3.2, we get the following result due to F. Skof [Skof, 1977].

Corollary 3.1: Let T be a self map of a complete metric space (X,d) and $\phi\in\Phi$ such that for every $x,y\in X$,

$$\phi\left(d(Tx,Ty)\right)\le\alpha\phi\left(d(x,Tx)\right)+\beta\phi\left(d(y,Ty)\right)+\gamma\phi\left(d(x,y)\right),$$

where $0\le\alpha+\beta+\gamma<1$. Then T has a unique fixed point.

If we take $\alpha=\beta,\delta=\theta=0,\phi(t)=t$ for every $t\ge 0$ in the theorem 3.2, we obtain the following result due to S. Reich [Reich, 1971].

Corollary 3.2: Let T be a self map of a complete metric space (X,d) and $\phi\in\Phi$ such that for every $x,y\in X$,

$$\phi\left(d(Tx,Ty)\right)\le\alpha\begin{pmatrix}\phi\left(d(x,Tx)\right)+\\ \phi\left(d(y,Ty)\right)\end{pmatrix}+\gamma\phi\left(d(x,y)\right),$$

where $0\le\alpha+\beta+\gamma<1$. Then T has a unique fixed point.

If we take $\alpha=\beta=\delta=\theta=0$ in the theorem 3.2, we obtain the following result.

Corollary 3.3: Let T be a self map of a complete metric space (X,d) and $\phi\in\Psi$ such that for every $x,y\in X,x\ne y$, $\phi\left(d(Tx,Ty)\right)\le\gamma\phi\left(d(x,y)\right)$, where $0\le\gamma<1$. Then T has a unique fixed point.

If $\phi(t)=t$ in the corollary 3.3, we obtain the following fixed point theorem of Rakotch [Rakotch, 1962].

Corollary 3.4: Let T be a self map of a complete metric space (X,d) and $\phi\in\Psi$ such that for every $x,y\in X,x\ne y$, $d(Tx,Ty)\le\gamma d(x,y)$, where $0\le\gamma<1$. Then T has a unique fixed point.

Conclusion: We have extended the theorems of Boyd, Wong [Boyd and Wong, 1969] and Matkowski [Matkowski, 1975] relaxing the upper semi-continuity of the function ψ in theorem 3.1. Further if we take α,β,δ and θ equal to 0 and ψ as identity function, we get the Banach Contraction mapping principle. Further we extended theorems of F. Skof [Skof, 1977], M. S. Khan et al. [Khan et. al., 1984], Rakotch [Rakotch, 1962] and Reich [Reich, 1971].

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